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Group Theory Structures Underlying Integrable Systems [1]

A.Mironov[2]

Theory Dept., Lebedev Physical Inst. and ITEP, Moscow, Russia

Different group structures which underline the integrable systems are considered. In some cases, the quantization of the integrable system can be provided with substituting groups by their quantum counterparts. However, some other group structures keep non-deformed in the course of quantizing the integrable system although their treatment is to be changed. Manifest examples of the KP/Toda hierarchy and the Liouville theory are considered.

1. During last decades there was a great development of integrable theories, both classical and quantum. However, while the classical integrable systems mostly advanced as the theory of non-linear equations, their quantum counterparts were rather based on algebraic structures related to the R-matrix. Therefore, the quantizing procedure was not always immediate. However, the last progress in integrable theories allows one to introduce a unified framework equally applicable to the both classical and quantum integrable systems. This framework is based on using the group theory structures which underline integrable system.

In fact, it was known for many years that the most elegant and effective approach to the classical integrable systems is to use the language of group theory [1]. However, it turns out that there are some *different* group structures underlying the same integrable system. Some of the groups act in the space of solutions to the integrable hierarchy, others can act just on the variables of equations ("in the space-time"). In order to quantize an integrable system, one needs just to replace the first type group structures by their quantum counterparts. It was demonstrated [2,3] that one can reformulate the classical non-linear equations in these group terms, i.e. the quantizing procedure becomes really immediate. On the other hand, the groups acting in the space-time still remain classical even for the quantum systems.

In this short remark we would like to stress the difference between above mentioned group structures and to demonstrate how they can be applied. Indeed, the groups acting in the space of solutions

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are used in the course of quantizing the system. At the same time, the space-time group can be used in the calculations of the two-point (quantum) correlation functions. The detailed discussion of the points briefly reviewed in this note is contained in [2,3,4].

2. At the first part, we demonstrate how a hierarchy of non-linear equations can be treated completely in group terms. This group acts in the space of solutions and should be replaced by the corresponding quantum group upon quantizing the system [2,3].

Let us start from the simplest example of the KP hierarchy. This hierarchy is an infinite set of equations for the infinitely many functions $\{u_i\}$ of infinitely many times $\{t_k\}$. The first equation of the hierarchy (it is the equation for the function $u_2(x,y,z)$ depending on the first three times $t_1 = x,\ t_2 = y,\ t_3 = z$) is the celebrated KP equation which has given the name for the whole hierarchy:

$$(u_{xxx} + 12uu_x - 4u_t)_x + 3u_{yy} = 0 \tag{1}$$

Other equations of the hierarchy have considerably more complicated form, more derivatives and more times involved. However, it was the striking invention of the Japanese school (Hirota [5] and Kyoto group [6]) that this hierarchy can be rewritten as an infinite set of equations for *the only* function of infinitely many times which is called τ-function. All the equations of the hierarchy are bilinear and homogeneous. The first one is

$$\left(D_1^4 + 3D_2^2 - 4D_1D_3\right)\tau\cdot\tau = 0 \tag{2}$$

where

$$u_2 = (\log\tau)_{xx}, \quad D_k^n f\cdot g \equiv \partial_\epsilon^k \left\{f(t_k+\epsilon)g(t_k-\epsilon)\right\}\Big|_{\epsilon=0} \tag{3}$$

Even more surprisingly, one can construct the general solution to the whole hierarchy in terms of the free fermionic system [6]. This solution is given by the ratio of the correlators

$$\tau(\{t_k\}) = \frac{\left\langle 0|e^{H(t)}g|0\right\rangle}{\langle 0|g|0\rangle}, \quad H(t) \equiv \sum H_k t_k \tag{4}$$

in the theory of free two-dimensional fermionic fields $\psi(z) = \sum_{\mathbf{Z}} \psi_n z^n dz^{\frac{1}{2}}$, $\psi^\star(z) = \sum_{\mathbf{Z}} \psi_n^\star z^{-n-1} dz^{\frac{1}{2}}$ with the action $\int \psi^\star\bar\partial\psi$. The Hamiltonians H_k giving the infinitely many commuting flows are defined as the positive modes of the current $J(z) \equiv \psi^\star(z)\psi(z) = \sum_k H_k z^{-k-1}$.

Element $g =\ :\exp\left\{\sum_{m,n} A_{mn}\psi_m^\star\psi_n\right\}:$ is an element of the group $GL(\infty)$ realized in the infinite-dimensional Grassmannian, and the normal-ordering should be understood with respect to the vacuum $|0>$. It was shown in [6] that the solutions to the KP hierarchy are in one-to-one correspondence with the different group elements g. Thus, we conclude that the group which acts in the space of solutions in the KP hierarchy case is $GL(\infty)$.

In fact, the same group acts in a little more general case of the two-dimensional Toda lattice hierarchy. This hierarchy is also satisfied by the only (τ-)function depending on *two* infinite sets of times $\{t_k\}$, $\{\bar{t}_k\}$ and one discrete index and is manifestly given by the ratio

$$\tau_n(t,\bar{t}|g) = \frac{\left\langle n|e^{H(t)}ge^{\bar{H}(\bar{t})}|n\right\rangle}{\langle n|g|n\rangle} \tag{5}$$

where the second set of times is coupled to the negative-mode (commuting) Hamiltonians $\bar{H}(\bar{t}) = \sum H_{-k}\bar{t}_k$ and vacuum states are defined by the conditions $\psi_m|k>=0,\quad m<k;\quad \psi^\star_m|k>=0,\quad m\geq k$. We observe that the space of solutions to the Toda hierarchy is still given by all the group elements of $GL(\infty)$. However, now we have more flows. Say, the first non-trivial equation is

$$\partial_1\tau_n\bar{\partial}_1\tau_n - \tau_n\partial_1\bar{\partial}_1\tau_n = \tau_{n+1}\tau_{n-1} \tag{6}$$

The next group which also has much to do with the space of solutions often arises as the group describing the reduction of the KP/Toda hierarchy. Two simplest examples are

1) $\widehat{SL(2)}$-reduction of KP, which corresponds to the KdV hierarchy (this hierarchy is described by the τ-function depending only on odd times, which means that, of all infinite set of independent functions u_i of the KP hierarchy, the only function u_2 is independent). The same reduction of the Toda hierarchy gives either Sine-Gordon or Toda chain hierarchies. Say, in the Toda chain case, this reduction means that the τ-function depends only on the difference of times $t_k - \bar{t}_k$. Then, making the substitute $e^{\phi_n} \equiv \frac{\tau_{n+1}}{\tau_n}$ in (6), one obtains the well-known Toda chain equation

$$\partial_1^2\phi_n = e^{\phi_{n+1}-\phi_n} - e^{\phi_n-\phi_{n-1}} \tag{7}$$

2) $SL(2)$-reduction of the Toda hierarchy. This reduction implies $\tau_{-n}=0$, $\tau_0=\tau_2=1$, $\tau_{n>2}=0$. Then, one gets $\partial_1\tau_1\bar{\partial}_1\tau_1 - \tau_1\partial_1\bar{\partial}_1\tau_1 = 1$ and, introducing $e^\phi \equiv e^{\phi_1-\phi_0} = \frac{1}{\tau_1^2}$, obtains the Liouville equation $\partial_1\bar{\partial}_1\phi = 2e^\phi$.

Therefore, say, the Liouville system is described by the group pair $(GL(\infty), SL(2))$ (see also [3]).

3. Now let us describe the general construction extending the KP/Toda hierarchies. For doing this, we stress the main two features of the considered hierarchies which are in charge of all their main properties. These features are

1) The element g is a group element (of $GL(\infty)$ in our case), i.e., put it differently, satisfies the co-product relation $\Delta(g) = g\otimes g$ if one looks at $GL(\infty)$ as Hopf algebra.

2) The fermions are transformed under the group action as

$$g\psi_i g^{-1} = R_{ik}\psi_k, \quad g\psi^\star_i g^{-1} = \psi^\star_k R^{-1}_{ki} \tag{8}$$

where R_{ik} is a numerical matrix (this can be immediately checked from the definition of g). This means that the fermions are intertwining operators which intertwine the fundamental representations of $GL(\infty)$. In fact, if one defines F_n to be the n-th fundamental representation,

$$\psi: \ F_1 \otimes F_n \to F_{n+1}, \qquad \psi^\star: \ F_{n+1} \to F_n \otimes F_1 \tag{9}$$

and index i of fermions runs over the first fundamental representation F_1.

Relation (8) implies that the combination $\sum_i \psi_i \otimes \psi_i^\star$ commutes with g. In turn, this results into the infinite set of (bilinear) identities for the τ-function forming the whole hierarchy of (KP/Toda) equation [6].

After this preparation, we are ready to consider the general case. Indeed, let be given a highest-weight representation λ of some Lie algebra $\mathcal{G}$ (indeed, we need the universal enveloping algebra $U(\mathcal{G})$ and *its* representations). Then, we define τ-function as the generating function of all matrix elements in the representation λ:

$$\tau^{(\lambda)}(t,\bar{t}|g) \equiv < 0| \prod_k e^{t_k T_-^{(k)}} \ g \ \prod_i e^{\bar{t}_i T_+^{(i)}} |0>_\lambda = \sum < \boldsymbol{n}|g|\boldsymbol{m}>_\lambda \prod_{i,j} \frac{t_i^{n_i} \bar{t}_j^{m_j}}{n_i! m_j!} \tag{10}$$

where the vacuum state means the highest weight vector, $T_\pm^{(k)}$ are the generators of the corresponding Borel subalgebras of $\mathcal{G}$ and the exponentials are supposed to be somehow normal-ordered ($\boldsymbol{n}$ ($\boldsymbol{m}$) is vector with the components n_i (m_i)).

This general τ-function (10) still satisfies some homogeneous bilinear identities. To demonstrate this [2], one needs to consider a triple of representations $V, \hat{V}, W$ which are intertwined by the operators

$$\Phi: \ \hat{V} \otimes W \to V, \qquad \Phi^\star: \ V \to W \otimes \hat{V} \tag{11}$$

Again there is a canonical element which commutes with the group element g. It is provided by the defining property of intertwiner which generalizes (8):

$$\Delta(g)\Phi = \Phi g, \quad \Phi^\star \Delta(g) = g \Phi^\star \tag{12}$$

4. Now let us say some words of quantizing this system [2,3]. One can replace the group by the corresponding quantum group and repeat the procedure of the previous section, but there are some new features to be pointed out:

1) τ-function is no longer commutative. It is still to be defined by formula (10), with exponential being replaced by the corresponding q-exponentials e_q.

2) One needs to differ between left and right intertwiners depending on the order of spaces in (11).

3) The counterpart of the group element g defined by the property $\Delta(g) = g \otimes g$ *does not* belong to the universal enveloping algebra $U_q(\mathcal{G})$ but to the product $U_q(\mathcal{G}) \otimes U_q^\star(\mathcal{G})$, where $U_q^\star(\mathcal{G}) \equiv A_q(G)$ is the dual (or, equivalently, the algebra of functions on the quantum group) [3].

To give some example of this last point, let us consider the group element in more explicit terms [7,3]. In the classical case, it can be manifestly parametrized as

$$g = \prod_{ijk} e^{x_-^{(i)} T_-^{(i)}} e^{x_0^{(j)} T_0^{(j)}} e^{x_+^{(k)} T_+^{(k)}} \tag{13}$$

where $x^{(i)}$ are the coordinates on the group manifold. In the quantum group case, these coordinates become non-commutative and generates the algebra of functions $A_q(G)$. (In the simplest $A_q(SL(2))$ case, their commutation relations are $[x_\pm, x_0] = (\log q)\, x_\pm,\ [x_+, x_-] = 0$.) While in the classical case all the representations of this algebra are trivial and are given just by the numerical values of the coordinates $x^{(i)}$, the representations of the quantum algebra are considerably more involved and, in particular, sometimes infinite-dimensional.

In the quantum case, the corresponding group element takes the form

$$g = \prod_{ijk} e_q^{x_-^{(i)} T_-^{(i)}} e^{x_0^{(j)} T_0^{(j)}} e_{q^{-1}}^{x_+^{(k)} T_+^{(k)}} \tag{14}$$

Since the τ-function (10) is the average of an element from $U_q(\mathcal{G}) \otimes A_q(G)$ over some representation of $U_q(\mathcal{G})$, it is an element of the algebra of functions $A_q(G)$, and, therefore, is non-commutative (see 1) above).

How we already discussed, in the classical case any concrete element g corresponds to some solution to the corresponding classical hierarchy. On the other hand, the element g is given by some (concrete) values of $x^{(i)}$, i.e. by some fixed (trivial) representations of the algebra of functions. This picture is completely extended to the quantum case: **representation of $A_q(G)$ corresponds to some solution to (quantum) hierarchy**. Certainly, any reduction, which selects out a subspace in the space of solutions, just restricts available representations and usually can be described by some additional group structure.

To conclude this section, let us note that, for the above introduced quantum group element g, there exists the natural group multiplication giving the (semi)-group structure [3]. Therefore, for the quantized hierarchy, the underlying (quantum) group still can be described as acting on the elements g, i.e. in the space of solutions to the (quantum) hierarchy.

5. Now let us consider some examples of the space-time group structure. For the lack of space, we consider only the quantum case (see [4] for the classical one and for the proper references). Let us start with the Liouville quantum mechanics and find the wave function of the Liouville Hamiltonian.

For doing this, we consider the group element of the *classical* $SL(2)$ group $g = e^{\psi T_-} e^{\phi T_0} e^{\chi T_+}$ and fix some irreducible spin j representation from the principal (spherical) series. We want it to be unitary, i.e. $j + \frac{1}{2}$ is pure imaginary. Now take the vectors from this representation satisfying the following properties

$$< \psi_L | T_- = i\mu_L < \psi_L |, \quad T_+ | \psi_R >= i\mu_R | \psi_R > \tag{15}$$

The second Casimir operator $C_2 = 2T_-T_+ + T_0 + \frac{1}{2}T_0^2 = 2j(j+1)$ in this representation. Now we consider the function $F(\phi) \equiv < \psi_L | e^{\phi T_0} | \psi_R >= e^{-i(\mu_L \psi + \mu_R \chi)} < \psi_L | g | \psi_R >$. It is called Whittaker function (see references in [4]). This function is proportional to the Liouville wave function:

$$\begin{aligned} 2j(j+1)F(\phi) &=< \psi_L | e^{\phi T_0} C_2 | \psi_R >=< \psi_L | e^{\phi T_0} \left(2T_-T_+ + T_0 + \frac{1}{2}T_0^2 \right) | \psi_R >= \\ &= \left(-2\mu_L \mu_R e^{-2\phi} + \frac{\partial}{\partial \phi} + \frac{1}{2}\frac{\partial^2}{\partial \phi^2} \right) F(\phi) \end{aligned} \tag{16}$$

i.e. $\Psi(\phi) \equiv e^{\phi} F(\phi)$ satisfies

$$\left(\frac{1}{2}\frac{\partial^2}{\partial \phi^2} - 2\mu_R \mu_L \right) \Psi(\phi) = 2\left(j + \frac{1}{2} \right)^2 \Psi(\phi) \tag{17}$$

Now to calculate this function, one can take the manifest realization of $SL(2)$ by the differential operators

$$T_+ = \frac{\partial}{\partial x}, \; T_0 = -2x\partial_x + 2j, \; T_- = -x^2 \partial_x + 2jx \tag{18}$$

and, solving differential equations (15), obtain

$$\begin{aligned} &\psi_R(x) = e^{i\mu_R x}, \quad \psi_L(x) = x^{-2(j+1)} e^{-\frac{i\mu_L}{x}}, \\ \text{i.e.} \quad &\Psi(\phi) = e^{\phi} \int dx x^{-2(j+1)} e^{-\frac{i\mu_L}{x}} e^{\phi\left(2j - 2x\frac{\partial}{\partial x}\right)} e^{i\mu_R x} \sim K_{2j+1}\left(2\sqrt{\mu_L \mu_R} e^{-\phi} \right) \end{aligned} \tag{19}$$

Using this integral representation, one can easily obtain the asymptotics of the function $\Psi(\phi)$:

$$\Psi(\phi) \underset{\phi \to \infty}{\sim} \frac{1}{\Gamma(1+\nu)} e^{\nu\phi} - \frac{1}{\Gamma(1-\nu)} e^{-\nu\phi}, \quad \nu \equiv 2j+1 \tag{20}$$

The functions $c_\pm(\nu) = \frac{1}{\Gamma(1\pm\nu)}$ giving these asymptotics are called Harish-Chandra functions. Their ratio gives the S-matrix and, equivalently, 2-point function of the theory $S = \frac{c_+}{c_-} = \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}$. [3]

The same procedure can be done for any finite-dimensional group giving [4]

$$c(\boldsymbol{\lambda}) = \prod_{\boldsymbol{\alpha} \in \Delta^+} \frac{1}{\Gamma(1 + \boldsymbol{\nu} \cdot \boldsymbol{\lambda})} \tag{21}$$

where the product runs over all the positive roots of the algebra. Moreover, it can be continued to the affine case which describes the Liouville $2d$ quantum field theory. The result is

$$c(\boldsymbol{\lambda}) = \prod_{n \geq 0} \Gamma^{-1}(p + n\tau) \prod_{n \geq 1} \Gamma^{-1}(n\tau)\Gamma^{-1}(1 - p + n\tau) \tag{22}$$

[3] We omit everywhere from the S-matrix a trivial factor depending on the cosmological constant μ.

This expression still requires a careful regularization, but all the infinite products cancel from the corresponding reflection S-matrix (2-point function)

$$S(p) = \frac{c(-p)}{c(p)} = \frac{\Gamma(1+p)\Gamma(1+\frac{p}{\tau})}{\Gamma(1-p)\Gamma(1-\frac{p}{\tau})} \tag{23}$$

This result is to be compared with the formulas for the 2-point functions obtained in papers [8] in a very different way[4].

To conclude, let us stress again that the space-time group structure discussed in this section, although being **classical** one, describes the wave functions of the **quantum** systems. The same groups also describe the corresponding classical systems, within a slightly different treatment (see [4] and references therein).

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[4]In their notations, $p = 2iP/b$ and $\tau = b^2$.